Inhomogeneous Continuity Equation With Application to Hamiltonian ODE

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This talk represents joint work with L. Chayes (UCLA) and W. Gangbo (Georgia Tech). We are mainly interested in Hamiltonian dynamics where mass may go to infinity in finite time. We will start with some preliminaries which explains the situation for the mass conserved case. Then we will present some work on one particular inhomogeneous continuity equation.

1. Continuity Equation I. The continuity equation is given as displayed, with ρ being a probability density and v a velocity. A simple computation shows that this equation expresses the fact that the change in the volume is due to the flux in and out of the boundary of the volume.

2. Continuity Equation II. In particular, we note that if the support of ρ is strictly contained in some volume V, then the total change of the mass in the volume is zero. For measures, by testing against functions of compact suppor, we have the appropriate weak formulation of the equation.

3. Lagrangian Description I. So far we have looked at the evolution from the perspective of the density or the measure. We can instead look at the trajectories of particles. More precisely, given a velocity field v_t, we can look at the associated flow equation, given as displayed. $X_t(x)$ then represents the position of the particle at time t which started at position x initially.

4. Lagrangian Description II. Further, if we define μ_t to be the pushforward of μ_0, then μ_t satisfies the continuity equation. Here we define Ψ to be φ along a characteristic. Now the first equality follows from the definition of pushforward; the second equality follows from the definition of Ψ; the third equality follows from Fubini's Theorem and the Fundamental Theorem of Calculus; and the final expression is zero since φ vanishes at 0 and T.

5. Wasserstein Distance. The pushforward map from the previous slide actually induces a map from the "manifold" of flow maps into the "manifold" of densities. Here ρ_0 is a fixed reference density (which we think of as the initial density). On the flow map manifold we have a flat Riemannian inner product whereas on the density manifold we have a Riemannian inner product which varies from point to point; the geometry (especially the induced distance) in the former case is easier to understand. The upshot, from Otto's paper, is that the map Π induces a distance on the density manifold which is the Wasserstein distance.

6. A.C. Curves and the Continuity Equation. More precisely, we consider the space of probability measures with bounded second moment equipped with the Wasserstein distance. For measures, the distance is given as the result of an optimization over transportation plans. From the book of Ambrosio, Gigli and Savaré we find the following correspondence theorem: Given an absolutely continuous curve in $\mathscr{P}_2$, there exists velocity fields v_t such that the continuity equation is satisfied and the metric derivative is the L^2 norm of v_t; conversely, if a curve in $\mathscr{P}_2$ solves the continuity equation, then it is absolutely continuous and the metric derivative is given as shown if and only if v_t lies in the tangent space as described. As a result we have another characterization for the Wasserstein distance and we can also identify the tangent space.

7. Hamiltonian ODE I. Now let's shift gears and discuss Hamiltonian dynamics. In the deterministic situation, we get a system of ODE's as shown. We can wonder what happens in the infinite dimensional case. That is, how do we describe the situation when we start with a measure. The answer has been provided by the work of Gangbo and Ambrosio and others in the works displayed. Here, in the mass conserved case (more clear later what we mean), we are given a Hamiltonian functional and we say μ_t is a Hamiltonian ODE if the appropriate continuity equation is solved, and v_t lie in the appropriate space.

8. Hamiltonian ODE II. Here we see an example of a Hamiltonian functional. We have the kinetic energy term, the potential given by Φ and an interaction term. In this case the gradient is easy to compute and is as displayed. One version of the theorem from the Gangbo/Ambrosio paper is if we start with a functional satisfying a linear growth condition, then there exists a solution to the Hamiltonian ODE and also conservation of energy.

9. Mass Reaching Infinity in Finite Time. Let's examine the linear growth condition a bit. Recall we are trying to solve the continuity equation with $v_t = \mathbb{J}\nabla\mathscr{H}(\mu_t)$ and also our characteristic equations. So the linear growth condition implies that $|X_t|$ can grow at most exponentially in time and this for example preserves compact support and second moment, etc. However, if we have a growth condition which is superlinear, then an explicit computation shows that there is a finite time at which a particle will reach infinity.

10. Regularization: Fade With Arc Length. As a regularization tool, we settled on studying a situation where the particle loses masses according to how much it has traveled. For simplicity we will consider a spatially homogeneous way of doing this, encoded by a single parameter ε.

11. Inhomogeneous Continuity Equation. The situation described leads to the continuity equation with a non–trivial right hand side equal to $\varepsilon|v_t|\mu_t$. Basically, given initial measure μ_0 and velocity field v_t, we do two things: 1) travel according to the characteristics given by v_t and 2) reduce the amount of mass by e to the minus ε times the arc–length. Since the distance traveled is always bounded by the arclength, and mass is being lost exponentially fast in arc–length, it is fairly easy to see that the dynamics encoded by this equation always preserves the exponential moment.

12. A Distance for Measures I. This leads us to the question of what kind of distance can we use for deficient measures. First we observe that if D_1 and D_2 are both distances, then the square root of D_1^2 plus D_2^2 is also a distance. So a simple distance we can consider is given by a mass difference term plus a Wasserstein distance term which is the Wasserstein distance between the uniformly renormalized versions of the measures. Here we see the (possible) relative positions of the various measures. Notice that in general μ_t^*, which is given by some velocity field v_t is not the same as $\overline{\mu}_t$, since the way we reduce the mass is spatially non–homogeneous.

13. A Distance for Measures II. This distance is fairly simple and the geodesics are very easy to describe. Basically if we want a geodesic between μ_0 and μ_1, then we first take a Wasserstein geodesic between $\overline{\mu}_0$ and $\overline{\mu}_1$ and reduce the mass linearly.

14. Continuity of Dynamics I. Next we can ask how commensurate is this distance with the dynamics we are interested in. The answer is not entirely. In this example we start with delta masses. The velocity field causes the second particle to travel some distance and return in some time interval h, so that μ_h assigns a smaller mass to the second particle. Now if we renormalize and compute the Wasserstein distance, we see that 1) we only get $\sqrt{h}$ and also we have a big factor of D representing the distance between the particles. We will be able to establish Hölder–1/2 continuity, but we do need some moment assumption.

15. Continuity of Dynamics II. We do indeed establish that the dynamics is Hölder–1/2 continuous. Notice we can now consider velocity field growing as any power. This estimate actually turns out to be enough.

16. Application to Hamiltonian ODE I. By using a discretization scheme, we can prove existence of solution to a Hamiltonian version of the the inhomogeneous equation. Here the dynamics really involves Hamiltonian dynamics plus reduction in mass according to arc–length. Further, we can also take ε to zero and retrieve limiting measures.

17. Application to Hamiltonian ODE II. Currently we are investigating the limiting measures. The goal would be to show some appropriate limiting measure satisfies the continuity equation and also retrieve some conservation laws along the lines of e.g., the mass at time t is the initial mass minus the mass carried off to infinity. There are further questions concerning whether we can/should consider different inhomogeneous equations and distances and also actual physical systems of relevance.